Circle Transformation

Consider a circle with as little context as possible, and call it C. It is the division of spaces, inside and outside, but is itself entirely exclusive of space. Its environment is featureless, so there is no reference to determine its size or orientation. There are two points on C called A and B. Their locations in C are further distinguished only with reference to each other, such as being on opposite sides or the same side of C. A straight line connects the points and is called AB. Its orientation has no reference to be distinguished by.

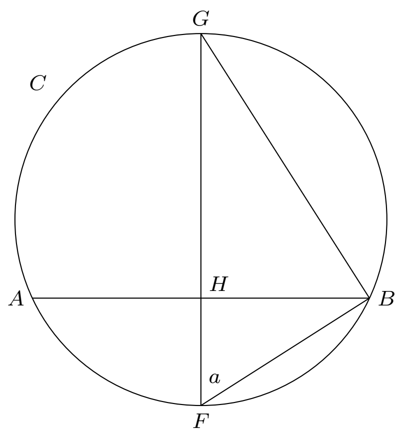
The viewer may assume that various relative locations of A and B correspond to various lengths of AB. If A and B occupy the same location, the length of AB is zero; if A and B are on opposite sides of C, length AB equals the diameter of C; and so on. Still, these various situations do not necessarily account for any difference in length. The assumption that various locations of A and B with respect to C correspond to various lengths of AB relies on the size of C as a constant reference, but in this context there is none.

By considering a circle, it is implied the viewer holds C within a frame of reference, demonstrating the inside is completely surrounded by the outside. This alone provides some reference to the size of C, because it is something C is always smaller than. The frame of reference is comparable to a shape D that circumscribes C.

Let D refer to the viewer's frame of reference. This introduces a kind of environment for A, B and C. Let C not be constrained to within D, but expand or contract to any size within or without it. Let A and B each have a constant, distinct location within D. If A and B are on opposite sides of C, then C appears at its smallest with respect to D (and the viewer). As A and B approach the same side of C, they don't change position with respect to D, but C expands, eventually past D, leaving only a segment visible to the viewer.

As the rest of C expands beyond the frame of reference, the curve of the circle between A and B appears more slight. The quality of curve between A and B is the true expression of their location with respect to C. Orientation is not included in this setting, and because the length of AB is constant, it is irrelevant to the various relative locations of A and B. All the viewer has left to recognize these different relative locations is the quality of curve that relates A and B.

Call H the midpoint of AB. Call FG a line that passes through H, is perpendicular to AB, and connects points F and G on circle C. The following geometric proof explains that lengths FH and GH are each the reciprocal of the other, if $BH = 1$.



Given:

1.	Circle C, points A, B, F, G, H
2.	Lines AB, FG, AH, BH, FH, GH, BF
3.	$AH = BH$
4.	$AB \perp FG$
5.	$\angle a$

Statements:		Reference:
1.	FG is the diameter of C.	Given 1 - 4, angle bisector theorem
2.	$\angle FBG$ is a right angle.	Given 1, Statement 1, Thale's theorem
3.	$\cos a = \frac{FH}{BF}$	Given 2, 4, 5
4.	$\cos a = \frac{BF}{FG}$	Given 2, 4, 5
5.	$\frac{FH}{BF} = \frac{BF}{FG}$	Statement 3, 4
6.	$FG = \frac{BF^2}{FH}$	Statement 5
7.	$FG = FH + GH$	Given 2
8.	$\frac{BF^2}{FH} = FH + GH$	Statement 6, 7
9.	$BF^2 = FH(FH + GH)$	Statement 8
10.	$BF^2 = BH^2 + FH^2$	Given 2, 4, Pythagorean theorem
11.	$BH^2 + FH^2 = FH^2 + FH(GH)$	Statement 9, 10
12.	$BH^2 = FH(GH)$	Statement 11

The set of all circles that pass through A and B form a continuous loop referred to as the circle tranformation. The animation that desribes it is generated by a computer program. The program sets a display window, and the location of AB is set with respect to it. The display is similar to D, except the circle can pass beyond it. Each frame in the animation displays one circle, and there are only so many frames, so the program determines a set of circles to represent the transformation. Similarly, a set of points is determined to represent each circle. Finally, the program determines the location of each point with respect to the display window. The set of circles, the set of points for each circle, and the location of each point are each determined by a mathematical equation. The equations rely on a reinterpretation of the scenario in terms of the Cartesian coordinate plane.

An equation to determine the (x, y) coordinates of any point on a given C is developed with reference to points A, B, F, and G, referred to as x_1, x_2, y_1, and y_2, respectively.

$x^2 + y^2 = r^2$	The development begins with this equation relating the x and y coordinates of any point of a circle to the circle's radius, provided the circle is centered on the origin of the Cartesian coordinate plane.
$x^2 + (y - h)^2 = r^2$	Every C is centered across the y axis, but the distance of the center from the x axis is different for each C. Variable h is introduced to allow for this difference, but needs to be reinterpreted in terms of constants.
$h = r + y_1$	The center's distance from the x axis is the length of the circle's radius minus any extra length that crosses the y axis. y_1 is already negative, so in this case the distance is the sum $r + y_1$.
$r = \frac{y_2 - y_1}{2}$	r is another variable that needs to be reinterpreted in terms of constants. The radius is half the diameter of a given C, or the sum of y_1 and y_2. Because y_1 is negative, the sum appears rather as a difference.
$r = \frac{1+y_1^2}{-2y_1}$	Although y_2 is constant, the computer program recognizes it only in terms of y_1. y_2 is replaced with $\frac{-1}{y_1}$, and the equation is simplified as follows:

$$\frac{\frac{-1}{y_1} - y_1}{2} \to \frac{\frac{-1}{y_1} - \frac{y_1^2}{y_1}}{2} \to \frac{\frac{-1-y_1^2}{y_1}}{2} \to \frac{-1 - y_1^2}{2y_1} \to \frac{1 + y_1^2}{-2y_1}$$

The new r identity is substituted in the equation $h = r + y_1$.

$$\frac{1 + y_1^2}{-2y_1} + y_1 \to \frac{1 + y_1^2}{-2y_1} - \frac{-2y_1^2}{-2y_1} \to \frac{1 - y_1^2}{-2y_1}$$

The new r and h identities are subsituted in the equation $x^2+(y-h)^2=r^2$.

$$x^2+\left(y-\frac{1-y_1^2}{-2y_1}\right)^2=\left(\frac{1+y_1^2}{-2y_1}\right)^2$$

The equation is modified to solve for x. One y input provides two x outputs, signified by the $\pm$ sign. They account for the left and right sides of the circle.

$$x=\pm\sqrt{\left(\frac{1+y_1^2}{-2y_1}\right)^2-\left(y+\frac{1-y_1^2}{2y_1}\right)^2}$$

For every circle, the equation is solved for a set of different y values, while the y_1 value stays the same, because it determines the circle being plotted. The set of y values for a given circle is determined by an equation, which ensures the points are evenly distributed. Call n the number of points on either side of the y axis, and i every interger from 1 to n.

$$y=y_1+\frac{y_2+y_1}{n}(i)$$

This equation provides one y value for every i. For every complete set of y values, the y_1 and n values stay the same, while the equation is solved for every possible i. In this case, the resulting points are evenly spread about the y axis, but when the x coordinates are applied, the points are not evenly spread about the circle.

$$y=h+r\times\sin\left(i\frac{\pi}{n}-\frac{\pi}{2}\right)$$

Provided the height and radius of the circle, this equation generates a set of y values that would be evenly distributed around the circle's circumference, if x coordinates were applied. The value inside the sine function is the interior angle of $\angle RST$, where S is at the center of the circle, R is at the circumference with the same y coordinate as S, and T is some other point along the same side of the circle's circumference. In radians, the range of $\angle RST$ is $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The absolute value of the range is divided into n sections and multiplied by some i, and then $\frac{\pi}{2}$ is subtracted to account for the negative half of the range.

Each cycle of the transformation is animated using the same set of y_1 values. For each cycle, y_1 ranges from $-\infty$ to 0, and y_2 ranges from 0 to ∞. Over two cycles, a point on C ranges from $-\infty$ to ∞, and is referred to as y_1 for one cycle and y_2 for the other cycle. The tangent function also ranges from $-\infty$ to ∞, while the domain is from 0 to π and every following cycle of π. The domain is much easier to partition into even sections than the range. By replacing y_1 with $\tan\theta$ in the equation for plotting points, the set of y_1 values become the set of θ values. The set of θ values is $\{\frac{\pi}{n}(i) : i = 1...n\}$. Replacing y_1 with θ is equivalent to replacing ∞ with π in this set, making it easier to compute.

$$x = \pm\sqrt{\left(\frac{1+\tan^2\theta}{-2\tan\theta}\right)^2 - \left(y + \frac{1-\tan^2\theta}{2\tan\theta}\right)^2}$$

For every cycle there is a y intercept at $-1 \leq y < 1$. This range is divided into equal sections easily, but the behavior of the resulting transformation is not so smooth. It is slow near $y = 0$ and fast near $y = -1$ and $y = 1$. The tangent function is already inherent to the transformation. The slopes of C at x_1 and x_2 both describe the tangent function. Throughout one cycle of the circle transformation, the slope of C ranges from $-\infty$ to ∞, and the angle of $\angle RST$ ranges from $-\pi$ to π. In this case S is at x_1 or x_2, R is a point on the x axis, and T is a point on C that is adjacent to x_1 or x_2.

The animation was generated using code written in the Processing language. Here the code is displayed from start to finish.

```
float z = (width / 2), w = 0, y, x, y1;
void settings() {
    size(700, 700);
}
void setup() {
    background(0);
    stroke(255);
}
void draw() {
    background(0);
    translate(width / 2, height / 2);
    w += PI / 1024;
    y1 = -tan(w);
    for (y = height / 2; y > -height / 2; y--) {
        x = z * sqrt(sq((1 + sq(y1)) / (-2 * y1))
        - sq((1 / z) * y + (1 - sq(y1)) / (2 * y1)));
        point(x, y);
        point(-x, y);
    }
}
```